

# A new eddy-viscosity model for large eddy simulation in helical turbulence

Changping. Yu<sup>11</sup>

<sup>1</sup>Peking University, Beijing 100871, China

(Dated: 13 December 2012)

In isotropic helical turbulence, a new single helical model is suggested for large eddy simulation. Based on the Kolmogorov's hypotheses, the helical model is proposed according to the balance of helicity dissipation and the average of helicity flux across the inertial range, and the helical model is a kind of eddy viscosity model. The coefficient of the helical model is constant and can be also determined dynamically. Numerical simulations of forced and decaying isotropic helical turbulence demonstrate that the helical model predicts the energy and helicity evolution well. The statistical character of the helical model is closer to the DNS results. In *a priori* test, the energy and helicity dissipations of the helical model also show a good results. In general, the helical model has the advantage contrast with the dynamic Smagorinsky and mixed models.

## I. INTRODUCTION

In turbulence flow, helicity is an important physical quantity. It is widespread in the motions of the atmosphere, ocean circulation and other natural phenomena, and also found in leading edge and trailing vortices shed from wings and slender bodies<sup>1-3</sup>. Helicity can be defined as,  $h = \mathbf{u} \cdot \boldsymbol{\omega}$ , where  $\mathbf{u}$  and  $\boldsymbol{\omega}$  are the velocity and vorticity of the turbulence flow respectively, and  $h$  is a pseudoscalar quantity. Similar with the status of energy in the dynamics of ideal fluids, helicity has the character of inviscid invariance. This physical property determines that helicity is an important quantity in turbulence research.

Recently, the researches on helical turbulence have developed greatly in theoretical research, experiment and numerical simulation. Based on the helical decomposition of velocity, the mechanism of existing a joint forward cascade of energy and helicity has been explained in theory<sup>4</sup>. Cascades existing in helical turbulence have space scale and time scale<sup>5-8</sup>, and the researches showed that the existing space scale of helicity cascade was large than energy cascade. In the inertial range, the joint cascade of energy and helicity was dominated by the energy cascade time scale in the low wave number and by the helicity cascade time scale in high wave number. Using direct numerical simulation of isotropic turbulence, energy and helicity flux were studied, the research showed that helicity flux is more intermittent than the energy flux and the spatial structure were much finer<sup>9</sup>.

Large eddy simulation (LES), as a important method, has been used to research helical turbulence. Several kinds of SGS models have been proposed so far, such as eddy-viscosity model, dynamic model, vortex model, subgrid-scale estimation model, and even the developing constrained SGS model<sup>10-23</sup>. Although so many types of SGS model, there are few models on account of the character of helical turbulence. Y. Li *et al.* proposed a two-term dynamic mixed models (DSH) to do large eddy simulations of helical turbulence<sup>24</sup>. Compared the LES results of the DSH model with other traditional models, the improvement of DSH model was not so remarkable. In this paper, we first use the balance of helicity

dissipation and helicity flux across the inertial range to get the eddy viscosity, then deduce a new helical model (SR). Using SR model and dynamic SR model (DSR) to do large eddy simulation of isotropic helical turbulence, we find that SR and DSR models can predict energy and helicity well. Contrast with dynamic Smagorinsky model (DSM) and DSH model, the new helical model is a simple model and its results of LES has get an obvious improvement.

## II. THEORETICAL ANALYSIS AND SGS MODEL CONSTRUCTION

In isotropic helical turbulence, the forced helicity control equation can be get from the N-S equation, as below

$$\partial_t h + \partial_j (u_j h) = \partial_j \Omega_j - 4\nu S_{ij} R_{ij} + 2f_i \omega_i, \quad (1)$$

where  $S_{ij} = \frac{1}{2}(\partial_j u_i + \partial_i u_j)$  is the strain rate tensor and  $R_{ij} = \frac{1}{2}(\partial_j \omega_i + \partial_i \omega_j)$  is the symmetric vorticity gradient tensor.  $f_i$  is the force, and  $\Omega_j$  is the flux term, which can be expressed as

$$\Omega_j = -\frac{p}{\rho} + \frac{1}{2}u_i u_i \omega_j + 2\nu(u_i R_{ij} + \omega_i S_{ij}) - \varepsilon_{jkl} u_k f_l. \quad (2)$$

For LES, the resolved helicity can be defined as  $h_\Delta = \tilde{\mathbf{u}} \cdot \tilde{\boldsymbol{\omega}}$ , and we can get the control function of  $h_\Delta$ <sup>24</sup> through filtering the Eq.(1) at scale  $\Delta$  as

$$\partial_t h_\Delta + \partial_j (\tilde{u}_j h_\Delta) = \partial_j \tilde{\Omega}_j - \Pi_H - 4\nu \tilde{S}_{ij} \tilde{R}_{ij} + 2\tilde{f}_i \tilde{\omega}_i, \quad (3)$$

where  $\Pi_H = -2\tau_{ij} \tilde{R}_{ij}$  is the SGS helicity dissipation rate, and  $\tau_{ij} = \tilde{u}_i \tilde{u}_j - \tilde{u}_i \tilde{u}_j$  is the SGS stress.

The equation for the resolved helicity in statistically stationary forced isotropic helical turbulence can be obtained through taking the ensemble average of Eq.(3) as

$$2\langle \tilde{f}_i \tilde{\omega}_i \rangle = -2\langle \tau_{ij} \tilde{R}_{ij} \rangle + 4\nu \langle \tilde{S}_{ij} \tilde{R}_{ij} \rangle. \quad (4)$$

where  $2\langle \tilde{f}_i \tilde{\omega}_i \rangle = \eta$  ( $\eta$  is the total helicity dissipation)  $-2\langle \tau_{ij} \tilde{R}_{ij} \rangle$  is the SGS helicity dissipation, and  $4\nu \langle \tilde{S}_{ij} \tilde{R}_{ij} \rangle$  is the viscous helicity dissipation at scale  $\Delta$ .

In the same way, we can also get the equation for the resolved energy in statistically stationary forced isotropic helical turbulence<sup>25</sup> from the filtered NS equation as

$$\langle \tilde{f}_i \tilde{u}_i \rangle = -\langle \tau_{ij} \tilde{S}_{ij} \rangle + 2\nu \langle \tilde{S}_{ij} \tilde{S}_{ij} \rangle. \quad (5)$$

where  $\langle \tilde{f}_i \tilde{u}_i \rangle = \epsilon$  ( $\epsilon$  is the total helicity dissipation),  $-\langle \tau_{ij} \tilde{S}_{ij} \rangle$  is the SGS energy dissipation and  $2\nu \langle \tilde{S}_{ij} \tilde{S}_{ij} \rangle$  is the viscous energy dissipation at scale  $\Delta$ .

From Eq.(4) and Eq.(5), we can see that the helicity dissipation has the similar form and composition as the energy dissipation.

In the inertial range of isotropic helical turbulence, the viscosity of the flow may be ignored, therefor we can get the helicity and energy relations<sup>26</sup>

$$\eta = -2\langle \tau_{ij} \tilde{R}_{ij} \rangle, \quad (6)$$

and

$$\epsilon = -\langle \tau_{ij} \tilde{S}_{ij} \rangle. \quad (7)$$

At the same time, the average of the energy flux across the inertial range is invariable and equals to the SGS energy dissipation,

$$\langle \Pi_E \rangle = -\langle \tau_{ij} \tilde{S}_{ij} \rangle, \quad (8)$$

where  $\Pi_E$  is the energy flux across the inertial range. Similar with energy, helicity has the same character in inertial subrange,

$$\langle \Pi_H \rangle = -2\langle \tau_{ij} \tilde{R}_{ij} \rangle, \quad (9)$$

where  $\Pi_H$  is the helicity flux across the inertial range.

In large eddy simulation of isotropic helical turbulence, we take the eddy-viscosity model as

$$\tau_{ij} - \frac{1}{3}\delta_{ij}\tau_{kk} = -2\nu_T \tilde{S}_{ij}, \quad (10)$$

where  $\nu_T$  is the eddy viscosity.

Based on the assumption of Eq.(9) and Eq.(10), we can obtain

$$\langle \Pi_H \rangle = -2\langle \tau_{ij}^{mod} \tilde{R}_{ij} \rangle = 2 \cdot \langle 2\nu_T \tilde{S}_{ij} \tilde{R}_{ij} \rangle, \quad (11)$$

where  $\tau_{ij}^{mod}$  is the SGS stress model.

In this deduction, we consider the eddy viscosity  $\nu_T$  as a global averaged value<sup>27</sup> for the time being, thus

$$\langle \Pi_H \rangle = 2\nu_T \langle 2\tilde{S}_{ij} \tilde{R}_{ij} \rangle. \quad (12)$$

In isotropic helical turbulence, we have

$$\langle 2\tilde{S}_{ij} \tilde{S}_{ij} \rangle = \int_0^{k_c} 2k^2 E(k) dk, \quad (13)$$

and

$$2\langle 2\tilde{S}_{ij} \tilde{R}_{ij} \rangle = \int_0^{k_c} 2k^2 H(k) dk, \quad (14)$$

where  $E(k)$  and  $H(k)$  are the energy and helicity spectra functions, and  $k_c$  is the cut off wavenumber.

The energy and helicity spectra  $E(k)$  and  $H(k)$  have the inequality<sup>23,24</sup> as

$$|H(k)| \leq 2kE(k). \quad (15)$$

In the inertial range, the energy and helicity spectra<sup>25</sup> can be choosed as

$$E(k) = C_K \epsilon^{2/3} k^{-5/3}, \quad (16)$$

and

$$H(k) = C_H \eta \epsilon^{-1/3} k^{-5/3}, \quad (17)$$

$C_K$  is Kolmogorov constant and  $C_H$  is the coefficient of the helicity spectrum.

Substituting Eq.(17) into Eq.(14), we can get

$$2\langle 2\tilde{S}_{ij} \tilde{R}_{ij} \rangle = \frac{3}{2} C_H \eta \epsilon^{-1/3} k_c^{4/3}. \quad (18)$$

From Eq.(15), we can see clearly that

$$\frac{|H(k)|}{E(k)} = \alpha k, \quad (19)$$

where  $\alpha$  is a dimensionless parameter, its value changes with  $k$ , and  $0 < \alpha \leq 2$ .

When the cut wavenumber is in the inertial range, from Eq.(16), Eq.(17) and Eq.(19), we obtain

$$\epsilon = \frac{C_H}{C_K} \alpha^{-1} k^{-1} |\eta|. \quad (20)$$

Taking the absolute value of both side of Eq.(18), and putting it into Eq.(20), we can get the expression

$$2|\langle 2\tilde{S}_{ij} \tilde{R}_{ij} \rangle| = \frac{3}{2} (\alpha C_K C_H^2)^{1/3} |\eta|^{2/3} k_c^{5/3}. \quad (21)$$

Then, from Eq.(6), (9), (11) and (21), we can obtain the expression of  $\nu_T$ ,

$$\nu_T = \frac{1}{2} C_r \Delta^{5/2} \tilde{S}_r, \quad (22)$$

where  $\Delta = \pi/k_c$  is the filter scale,  $\tilde{S}_r = |\langle 2\tilde{S}_{ij} \tilde{R}_{ij} \rangle|^{1/2}$  and the model coefficient  $C_r = (\frac{4}{3})^{3/2} \pi^{-5/2} (\alpha C_K C_H^2)^{-1/2}$ . Thus, we obtain a new helical SGS model as

$$\tau_{ij}^{mod} = -C_r \Delta^{5/2} \tilde{S}_r \tilde{S}_{ij}. \quad (23)$$

In the LES model,  $\alpha$  is set to 2.0 and  $C_K = 1.6$  and  $C_H$  is confirmed by the DNS data about 1.35.

In this letter, the eddy viscosity  $\nu_T$  of the model may be given a local value, and the expression of  $\nu_T$  is

$$\nu_T = \frac{1}{2} C_r \Delta^{5/2} \tilde{S}_{SR}, \quad (24)$$

where  $\tilde{S}_{SR} = |2\tilde{S}_{ij} \tilde{R}_{ij}|^{1/2}$ . Thus we get the new helical model discussed in this letter (SR model) as

$$\tau_{ij}^{mod} = -C_r \Delta^{5/2} \tilde{S}_{SR} \tilde{S}_{ij}. \quad (25)$$

where  $C_r \approx 0.036$ . The coefficient of helical model  $C_r$  can also be decided dynamically, and then we get the DSR model in this letter.

### III. THE NUMERICAL RESULTS AND ANALYSIS

In this part, we will give *a priori* and *a posteriori* test of the LES model, and do some comparison and analysis.

In order to validate our model, a DNS of three-dimensional incompressible homogeneous isotropic turbulence is introduced here. It solves the forced N-S equations using a pseudo spectral code in a cubic box with periodic boundary conditions, and the numerical resolution is  $512^3$ . A Gaussian random field is the initial flow condition, and it has an energy spectrum as

$$E_0(k) = Ak^2 U_0^2 k_0^{-5} e^{-\frac{2k^2}{k_0^2}}, \quad (26)$$

where  $k_0 = 4.5786$  and  $U_0 = 0.715$ . The whole system is maintained by a constant energy input rate  $\epsilon = 0.1$  and  $\eta = 0.3$  in the first two wave number shells.

Have constructed the new helical models SR and DSR, we first test the validity of the models *a priori*. In the inertial range, we have assumed the invariance of the energy and helicity SGS dissipations. In such a precondition, the invariance of  $f(\delta)$  and  $h(\delta)$  must hold,

$$f(\delta) = \langle \delta^{2.5} \tilde{S}_{SR} \tilde{S}_{ij} \tilde{S}_{ij} \rangle, \quad (27)$$

and

$$h(\delta) = \langle \delta^{2.5} \tilde{S}_{SR} \tilde{S}_{ij} \tilde{R}_{ij} \rangle, \quad (28)$$

where  $\delta$  is the length scale varying in the inertial range.

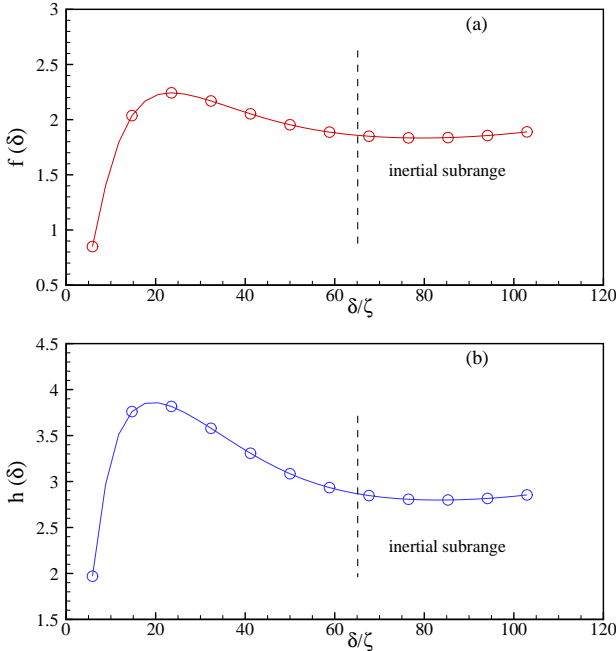


FIG. 1. (a)  $f(\delta)$ , (b)  $h(\delta)$  distribute with  $\delta/\zeta$  for *a priori*.  $\zeta$  is the Kolmogorov length scale.

In Fig.1, we show  $f(\delta)$  and  $h(\delta)$  as a function of  $\delta/\zeta$ , and  $\zeta$  is the Kolmogorov scale. It is easy to see that the

values of the two functions in inertial range are almost constant, and it demonstrates the scale-invariance and the reasonableness of the model.

We choose four LES models to do some analysis and comparison here, the SR model, the DSR model, the dynamic Smogorinsky model (DSM) and the dynamic mixed helical model (DSH)<sup>21</sup>.

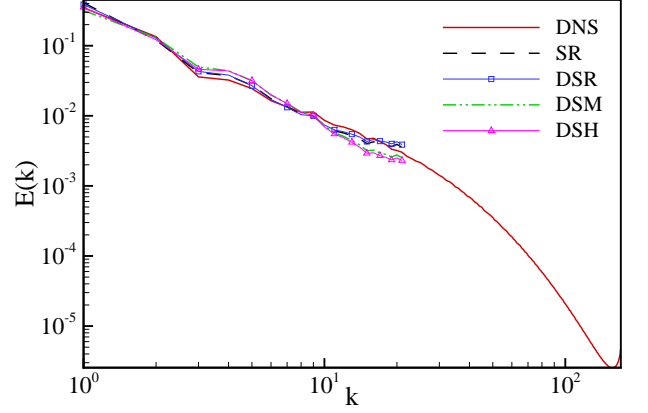


FIG. 2. Energy spectra for steady isotropic turbulence. Solid line: DNS; dashed line: SR; line with square: DSR; dash-dot-dotted line: DSM; line with delta: DSH.

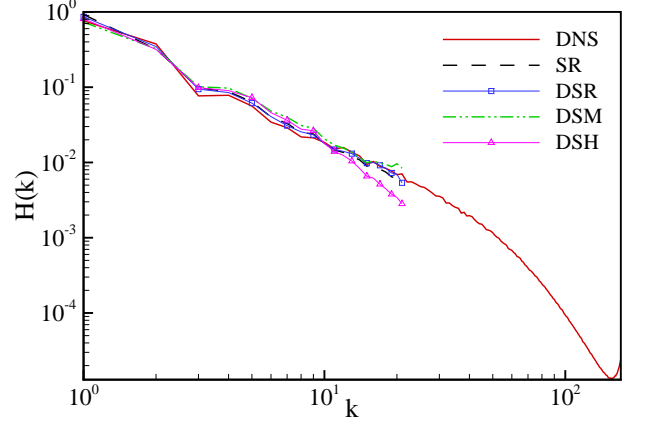


FIG. 3. Helicity spectra for steady isotropic turbulence. Solid line: DNS; dashed line: SR; line with square: DSR; dash-dot-dotted line: DSM; line with delta: DSH.

Fig.2 and Fig.3 display the energy and helicity spectra of DNS and different models for steady isotropic turbulence respectively. The bolt solid line is for DNS spectra, and the other lines are for the four models. It is easy to see that the SR and DSR show similar trends and predict the energy and helicity spectra better than DSM and DSH. In Fig.2, DSM and DSH underestimate the energy spectra near the cutoff wave-number and overestimate at the middle range of the wave-number. In Fig.3, DSM and DSH overestimate the helicity spectra at the middle range of the wave-number, and DSH underestimates it seriously near the cutoff wave-number.

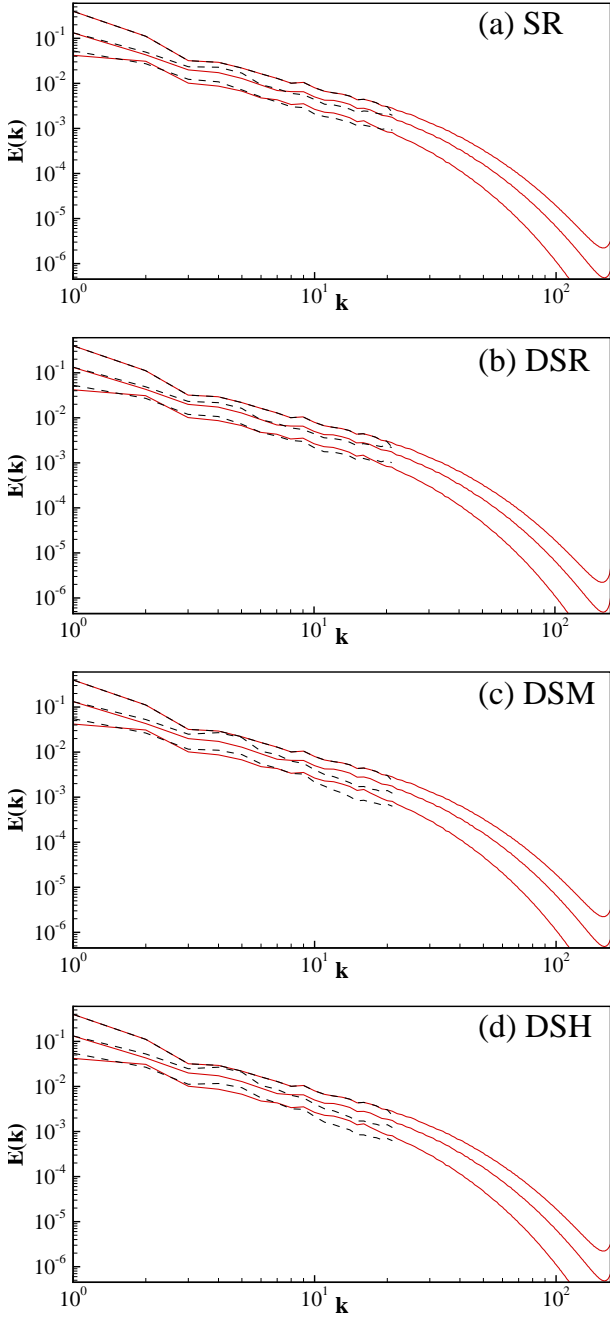


FIG. 4. Energy spectra for decaying isotropic turbulence (*a posteriori*), at  $t = 0, 6\tau_0$ , and  $12\tau_0$ , where  $\tau_0$  is the initial large eddy turnover time scale. Solid line: DNS; dashed line: (a) SR, (b) DSR, (c) DSM, (d) DSH.

In Fig.4 and Fig.5 we show the time evolution of the DNS and four models' energy and helicity spectra for a decay problem starting from a fully developed statistical steady state respectively, where  $t = 0, 6\tau_0$  and  $12\tau_0$ . In Fig.4, similar with the steady problem, the SR and DSR also show similar trends and predict the energy spectra very well. DSM and DSH overestimate the energy spectra at the middle range of the wave-number and underesti-

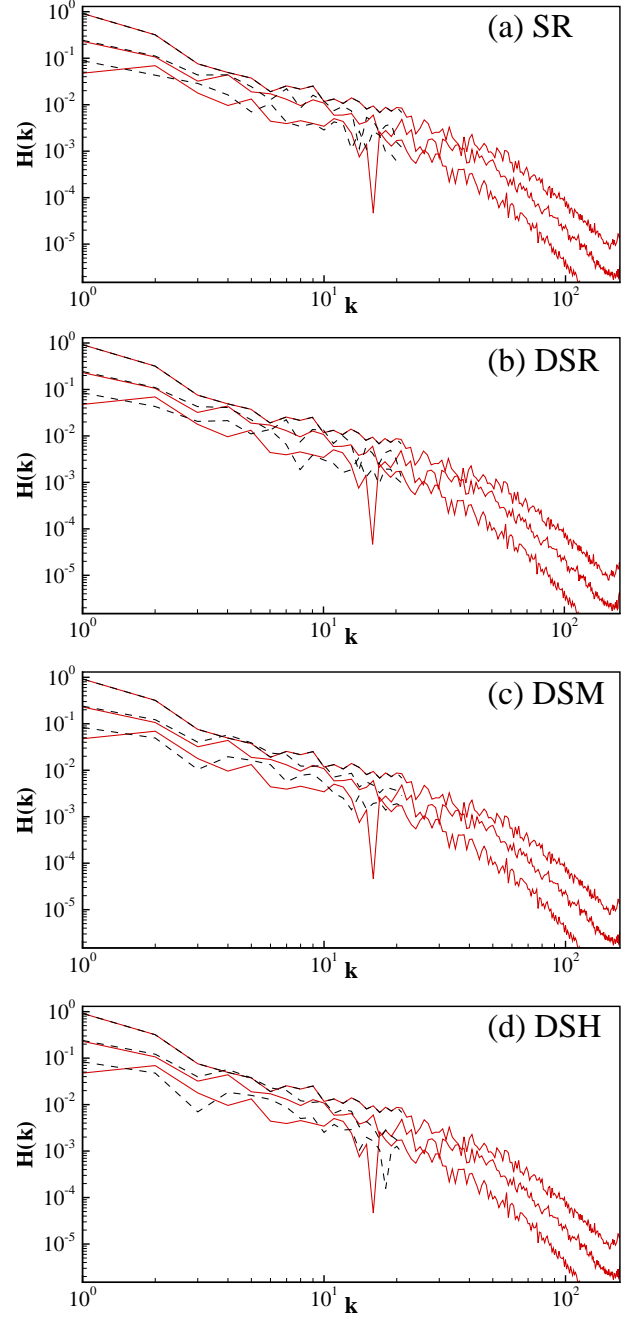


FIG. 5. Helicity spectra for decaying isotropic turbulence (*a posteriori*), at  $t = 0, 6\tau_0$ , and  $12\tau_0$ , where  $\tau_0$  is the initial large eddy turnover time scale.. Solid line: DNS; dashed line: (a) SR, (b) DSR, (c) DSM, (d) DSH.

mate it near the cutoff wave-number. Fig.5 show us the graphics different from Fig.3 greatly. The helicity spectra have large fluctuation, because it is a free decaying course and helicity is a pseudoscalar quantity, the two factors cause such phenomenon. In Fig.5 we can still see SR and DSR predict the helicity spectra a little better than the other two models.

Fig.6 and Fig.7 show us the SGS energy and helicity

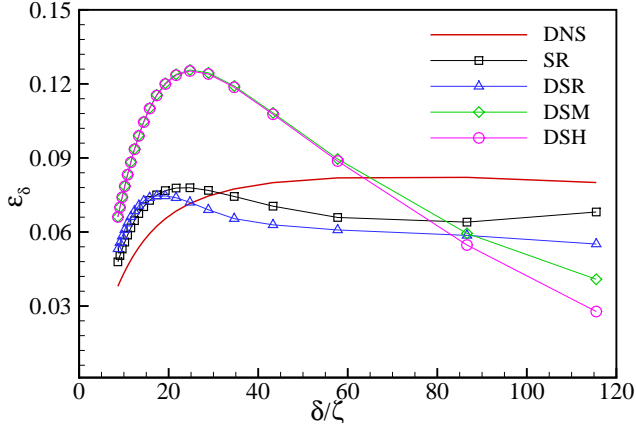


FIG. 6. The SGS energy dissipation (steady) distribute with  $\delta/\zeta$  for *a priori*. Bold solid line: DNS; line with Square: SR; line with delta: DSR; line with diamond: DSM; line with circle: DSH.

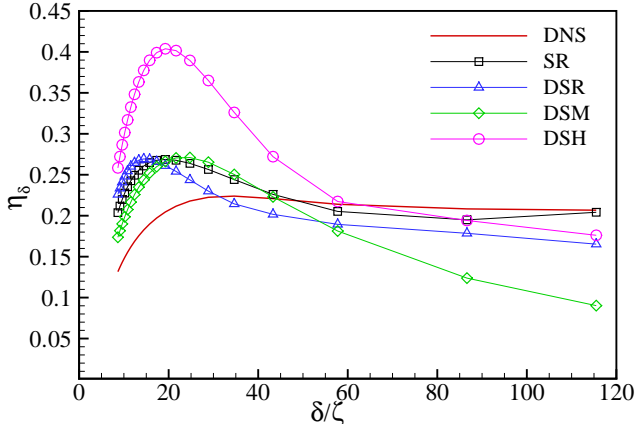


FIG. 7. The SGS helicity dissipation (steady) distribute with  $\delta/\zeta$  for *a priori*. Bold solid line: DNS; line with Square: SR; line with delta: DSR; line with diamond: DSM; line with circle: DSH.

dissipations of DNS and the four models distribute with  $\delta/\zeta$  for *a priori* respectively, and they reflect case of the full developed steady turbulence. It is obvious from in the inertial range that the SGS energy and helicity dissipations from SR are closer to those of the DNS than other models, and specially have the similar change trends with DNS.

In Fig.8 and Fig.9, we show the decay of SGS energy and helicity dissipation with  $t/\tau_0$  ( $\tau_0$  is the initial large eddy turnover time scale) from a fully developed steady state for *a priori* at the filter scale  $\Delta$  respectively. It is clear in Fig.8 that the SGS energy dissipations from SR and DSR are closer to the DNS result than DSM and DSH, and also have similar change trend. And also we can see in Fig.9 that the SGS helicity dissipations from SR, DSR and DSM are closer to the DNS result than DSH.

In Fig.10, we display the distribution of high-order ve-

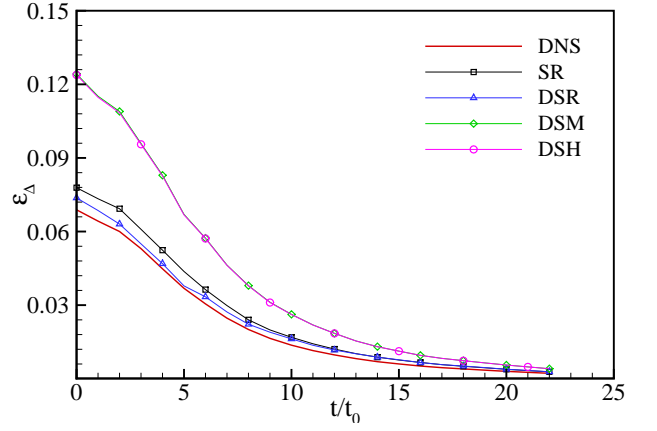


FIG. 8. The decay of SGS energy dissipation with  $t/\tau_0$  ( $\tau_0$  is the initial large eddy turnover time scale) from a fully developed steady state for *a priori* at the filter scale  $\Delta$ . Bold solid line: DNS; line with Square: SR; line with delta: DSR; line with diamond: DSM; line with circle: DSH.

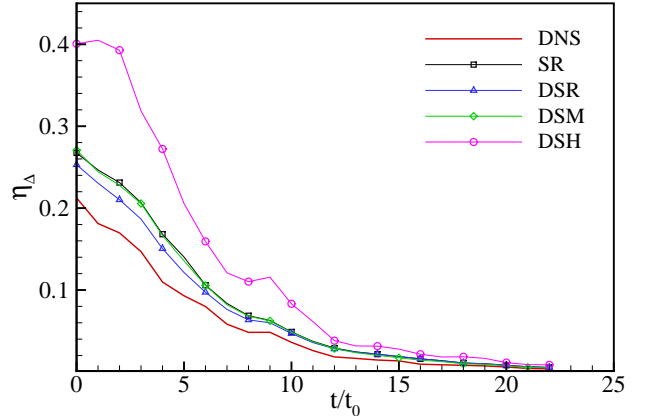


FIG. 9. The decay of SGS helicity dissipation with  $t/\tau_0$  from a fully developed steady state for *a priori* at the filter scale  $\Delta$ . Bold solid line: DNS; line with Square: SR; line with delta: DSR; line with diamond: DSM; line with circle: DSH.

locity increment with  $\Delta$ , where  $r$  is the separation distance and  $\Delta$  is the filter scale. Fig.10 (a), (b) and (c) denote the fourth-order, sixth-order and eighth-order structure functions of the longitudinal velocity increment of DNS and the four models distribute with separation distance. We can from Fig.10 that the result from SR is closest to the DNS result very well in the range of  $r/\Delta \geq 3$ , and the behavior of DSR is a little better than DSH and DSM. While in the range of  $r/\Delta < 3$ , anyone of the models have still not given a rather good result.

#### IV. CONCLUSIONS

In this letter, we construct a new helical SGS stress model for large eddy simulation of the isotropic helical turbulence. Different other eddy viscosity model, we con-



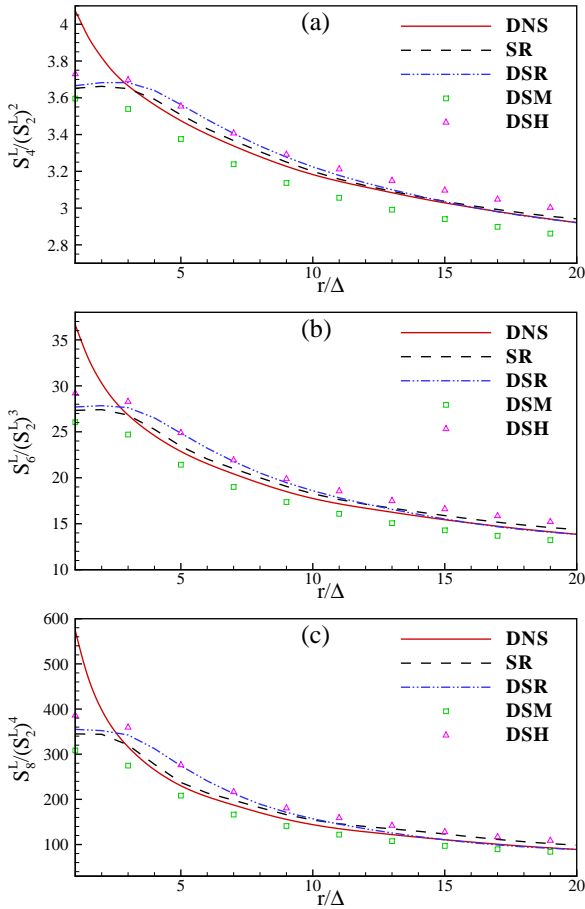


FIG. 10. The distribution of high-order moments of longitudinal velocity increment with the separation distance  $r$ , where  $\Delta$  is the filter scale. (a)  $S_4^L / (S_2^L)^2$ , (b)  $S_6^L / (S_2^L)^3$ , (c)  $S_8^L / (S_2^L)^4$

duct the helical model based on the balance of helicity dissipation and the average of helicity flux across the inertial range in helical turbulence. Then we have tested the scale-invariance of the model for *a priori* in inertial range, and make sure the validity of the helical model. We have given a constant coefficient helical model and a dynamic helical model here.

Have tested from *a priori* and *a posteriori*, the helical model is confirmed to predict energy and helicity spectra accurately, and also gives rather good simulating results in energy and helicity dissipation, *et al.* Besides the advantages above all, the helical model is a single model and can improve the computing efficiency greatly. As a SGS model on account of helical turbulence, it can also apply to the rotational turbulence.

## V. ACKNOWLEDGEMENT

This work was supported by the National Natural Science Foundation of China (Grant Nos. !!!!!!!).

- <sup>1</sup>D. K. Lilly, *The development and maintenance of rotation in convective storms* In *Intense atmospheric vortices: proceedings of the joint symposium (IUTAM/IUGG) held at Reading*, edited by L. Bengtsson and J. Lighthill (Springer-Verlag, Beilin, 1982), pp. 149-60.
- <sup>2</sup>A. Tsinober and E. Levich, "On the helical nature of three dimensional coherent structures in turbulent flows," *Phys. Rev. Lett.* **99A**, 321 (1983).
- <sup>3</sup>H. K. Moffatt and A. Tsinober, "Helicity in laminar and turbulent flow," *Annu. Rev. Fluid Mech.* **24**, 281 (1992).
- <sup>4</sup>Q. Chen, S. Chen, and G. L. Eyink, "The joint cascade of energy and helicity in three-dimensional turbulence," *Phys. Fluids* **15**, 361 (2003).
- <sup>5</sup>P. D. Ditlevsen and P. Giuliani, "Cascades in helical turbulence," *Phys. Rev. E* **63**, 036304 (2001).
- <sup>6</sup>P. D. Ditlevsen and P. Giuliani, "Dissipation in helical turbulence," *Phys. Fluids* **13**, 3508 (2001).
- <sup>7</sup>R. H. Kraichnan, "Inertial-range transfer in two- and three-dimensional turbulence," *J. Fluid Mech.* **47**, 525 (1971).
- <sup>8</sup>S. Kurien, M. A. Taylor, and T. Matsumoto, "Cascade time scales for energy and helicity in homogeneous isotropic turbulence," *Phys. Rev. E* **69**, 066313 (2004).
- <sup>9</sup>Q. N. Chen, S. Y. Chen, G. L. Eyink, and D. D. Holm, "Intermittency in the joint cascade of energy and helicity," *Phys. Rev. Lett.* **90**, 214503 (2003).
- <sup>10</sup>J. W. Deardorff, "The use of subgrid transport equation in a three dimensional model of atmospheric turbulence," *ASME*. **26**(6), 669 (1973).
- <sup>11</sup>S. B. Pope, *Turbulence flow* (Cambridge University Press, Cambridge, 2000).
- <sup>12</sup>J. Smagorinsky, "General circulation experiments with primitive equation," *Monthly Weather Review*, **91**, 99 (1963).
- <sup>13</sup>P. Moin and J. Kim, "Numerical investigation of turbulent channel flow," *J. Fluid Mech.* **118**, 341 (1982).
- <sup>14</sup>M. Germano, U. Piomelli, P. Moin, and W. Cabot, "A dynamic subgrid-scale eddy viscosity model," *Phys. Fluids A* **3**, 1760 (1991).
- <sup>15</sup>D. K. Lilly, "A proposed modification of the Germano subgrid-scale closure method," *Phys. Fluids A*, **4**, 3 (1992).
- <sup>16</sup>A. Misra and D. I. Pullin, "A vortex-based subgrid stress model for large eddy simulation," *Phys. Fluids* **9**, 2443 (1997).
- <sup>17</sup>J. A. Domaradzki and E. M. Saiki, "A subgrid-scale model based on the estimation of unresolved scales of turbulence," **9**, 2148 (1997).
- <sup>18</sup>J. A. Langford and R. D. Moser, "Optimal LES formulations for isotropic turbulence," *J. Fluid Mech.* **398**, 321 (1999).
- <sup>19</sup>Y. Mornishi and O. V. Vasilyev, "A recommended modification to the dynamic two-parameter mixed subgrid scale model for large eddy simulation of wall bounded turbulent flow," *Phys. Fluids* **13**, 3400 (2001).
- <sup>20</sup>C. Meneveau, "Statistics of turbulence subgrid-scale stresses; Necessary conditions and experimental tests," *Phys. Fluids* **6**, 815 (1994).
- <sup>21</sup>S. Ghosal, T. S. Lund, P. Moin, and K. Akselvoll, "A dynamic localization model for large-eddy simulation of turbulent flows," *J. Fluid Mech.* **286**, 229 (1995).
- <sup>22</sup>Y. Li and C. Meneveau, "Analysis of mean momentum flux in subgrid models of turbulence," *Phys. Fluids* **6**, 3483 (2004).
- <sup>23</sup>Y. Shi, Z. Xiao, and S. Chen, "Constrained subgrid-scale stress model for large eddy simulation," *Phys. Fluid* **20**, 011701 (2008).
- <sup>24</sup>Y. Li, C. Meneveau, S. Chen, and G. L. Eyink, "Subgrid-scale modeling of helicity and energy dissipation in helical turbulence," *Phys. Rev. E* **74**, 026 (2006).
- <sup>25</sup>C. Meneveau and T. S. Lund, "The dynamic Smagorinsky model and scale-dependent coefficients in the viscous range of turbulence," *Phys. Fluid* **9**, 3932 (1997).
- <sup>26</sup>D. k. Lilly, "The representation of small-scale turbulence in numerical simulation experiments," *Proceedings of the IBM Scientific Computing Symposium on Environmental Sciences*, 1967, p. 195.

- <sup>27</sup>S. Cerutti and C. Meneveau, "Intermittency and relative scaling of subgrid-scale energy dissipation in isotropic turbulence," *Phys. Fluid* **10**, 928 (1998).
- <sup>28</sup>A. Brissaud, U. Frisch, J. Leorat, M. Lesieur, and A. Mazure, "helicity cascades in fully developed isotropic turbulence," *Phys. Fluids* **16**, 1366 (1973)
- <sup>29</sup>V. Borue and S. A. Orszag, "spectra in helical three-dimensional homogeneous isotropic turbulence," *Phys. Rev. E* **55**, 7005 (1997).
- <sup>30</sup>Y. Zang, R. L. Street, and J. R. Koseff, "A dynamic mixed subgrid-scale model and its application to turbulent recirculating flows," *Phys. Fluids A* **5**, 3186 (1993).
- <sup>31</sup>S. W. Liu, C. Meneveau, and J. Katz, "On properties of similarity subgrid-scale models as deduced from measurements in a turbulent jet," *J. Fluid Mech.* **275**, 83 (1994).
- <sup>32</sup>C. Meneveau and J. Katz, "Scale-Invariance and turbulence models for large-eddy simulation," *Annu. Rev. Fluid Mech.* **32**, 1 (2000).
- <sup>33</sup>H. Lu, C. J. Rutland, and L. M. Smith, "A priori tests of one-equation LES modeling of rotating turbulence," *J. Turbulence*. **8**, 37 (2007).
- <sup>34</sup>Q. Chen and G. L. Eyink, "The joint cascade of energy and helicity in three-dimensional turbulence," *Phys. Fluid* **15**, 361 (2003).
- <sup>35</sup>C. Meneveau and J. Katz, "Scale-invariance and turbulence models for large-eddy simulation," *Annu. Rev. Fluid Mech.* **32**, 1 (2000).